jnh366 November 2021

Mathematics Analysis and Approach Standard Level Exploration.

Modelling an equation for a cooling cup of tea.

1. Introduction.

It has been more than 5 years that I've noticed that I had difficulties digesting food. At first, my stomach was bloated, and it was uncomfortable. I never really paid attention to it as I told myself that it might be ok after some days (Africans' mentality), but it wasn't the case. After some advice from my aunt who is dietician, I started drinking tea which helped me by facilitating my digestion. Today, tea is part of my life and I drink it every day, not because I want but because I must in order to digest well.

As an IB student, it happened more than once that I do my tea, and leave it to cool down to a certain temperature before drinking it, but at the end, I forgot about it because of work (you know how IB can be hectic sometimes) and it becomes too cold for me to drink it hence, I need to warm it (this annoys me). Because of this, I told myself that it could be interesting if I could have an alarm that rings and remind me exactly at what time my tea is suitable to drink (at my desired temperature).

I started thinking about it but never knew that it could be my Mathematics IA topic till the day I found a similar topic on the internet. From there, I started reading to know how this could be made possible. I was enlightened when we started the Math topic: Calculus where I finally understood that for this math experience to be done, I should apply my understanding in calculus alongside with other knowledge from functions, statistics and algebra.

By conducting this exploration, I aim to model an equation for the cooling of my cup of tea. This equation will help me to accurately calculate after what amount of time my cup of tea will have reached the desired temperature for me to drink it (as I prefer hot or warm tea than cold tea). Also, this will enable me to get a better understanding of the math topics that we did in class, especially calculus as applying it to a real-life situation helps me to have a concrete idea of its utility in our daily life.

2. Carrying out the experiment.

Materials.

For tea:

- 5 tea sachets.
- Water.
- Honey.

Tools:

- A mug.
- A spoon.
- An electric kettle.
- A thermometer.
- A stopwatch.

Method.

- Make a normal cup of tea with hot water.
- Remove the tea sachet.
- Add one teaspoon of honey.
- Stir completely.
- Note down the room temperature.
- Put the thermometer in the cup of tea and note down the initial tea temperature.
- Note down the temperature of the tea after every 5 minutes for two hours.

3. Data Collection.

The tables below show the temperature that I recorded for my cooling cup of tea over 2 hours in a series of five minutes intervals.

3.1 First Trial.

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.												
Temp (℃)	68.0	66.0	56.0	52.0	49.0	46.0	44.5	41.5	40.0	38.0	37.0	36.0
Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.													
Temp (℃)	34.5	33.5	33.0	32.0	31.0	30.5	30.0	29.5	29.0	29.0	28.5	28.0	27.5
Time (mins)	60	65	70	75	80	85	90	95	100	105	110	115	120

Table 1.0: Tables of values indicating the first trial of results.

3.2 Second Trial.

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.												
Temp (℃)	67.0	60.0	56.0	52.0	48.5	45.5	44.0	41.5	39.5	38.0	36.5	35.5
Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.													
Temp (℃)	34.5	33.5	32.5	32.0	31.5	31.0	30.0	29.5	29.0	29.0	28.0	28.0	27.5
Time (mins)	60	65	70	75	80	85	90	95	100	105	110	115	120

Table 1.1: Tables of values indicating the second trial of results.

3.3 Third Trial.

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.												
Temp (℃)	69.0	64.0	59.5	56.0	53.0	50.0	48.0	45.0	43.5	41.5	40.0	39.0
Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.													
Temp (℃)	37.5	36.5	35.5	34.5	34.0	33.0	32.0	31.5	31.0	30.5	30.0	29.5	29.0
Time (mins)	60	65	70	75	80	85	90	95	100	105	110	115	120

Table 1.2: Tables of values indicating the third trial of results.

Pictures of the experiment.

Picture 1: Materials and tools needed for the experiment.

Picture 2: Process of measuring the temperature for a cooling cup of tea.

4. Calculating the mean values for the temperature recorded for each time interval.

I decided to do three trials of the experiment to get three sets of values from which I will calculate the mean temperature value for each time interval. This helps me to eliminate uncertainties and to increase the accuracy of my results.

Formula: $\boldsymbol{\bar{X}} = \frac{\sum(\mathbf{x})}{\boldsymbol{n}}$

Where:
$\bar{X}$ indicates the mean.
$\sum$(x) indicates the sum of data values.
n indicates the total number of data values.

In other words, formula: $\frac{\textit{Sum of all values under the same time interval}}{\textit{Number of values under the same time interval}}$

The table below shows the mean temperature for the three sets of values.

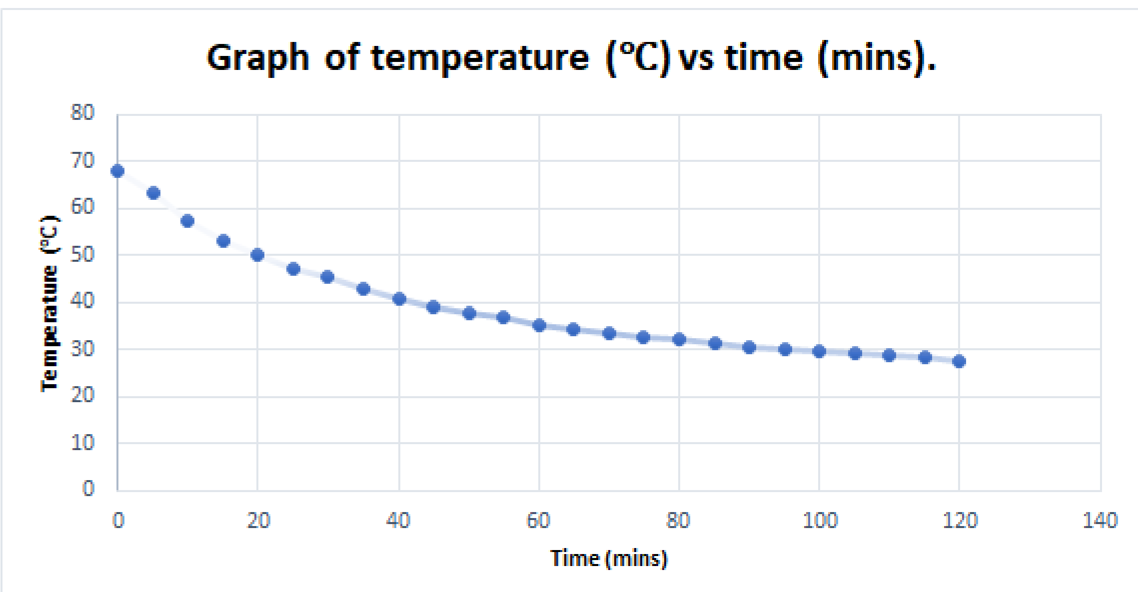
Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.												
Temp (℃)	68.0	63.3	57.2	53.3	50.2	47.2	45.5	42.7	40.8	39.2	37.8	36.8
Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.													
Temp (℃)	35.3	34.3	33.7	32.8	32.2	31.3	30.7	30.0	29.7	29.5	28.8	28.5	27.8
Time (mins)	60	65	70	75	80	85	90	95	100	105	110	115	120

Table 1.3: Tables of mean values for all the three sets of values.

5. Graphic representation.

I used Microsoft Excel to produce a graph of the mean temperature values of the cooling tea with temperature at the y-axis and time at the x-axis.



Graph 1.0: Graph of temperature vs time of the cooling cup of tea.

6. Analyzing the graph above.

The graph shows an exponential curve from which we can assume the following:

1. The tea temperature will never go below the room temperature (the environment in which the tea is) which is 24.0℃ because it is the difference between the tea temperature and the room temperature that makes the tea cool down. Hence, 24.0℃ is the horizontal asymptote.
2. As time goes, the tea temperature decreases. This gives the graph a negative correlation. Because of this, the proportionality constant of the graph will be negative: -k
3. When time = 0 (x-axis), the initial temperature of the tea is 68.0℃ (y-axis). Hence, it is the y-intercept.

7. Producing an equation to model the cooling of a cup of tea based on Newton's Law of Cooling and the previous assumptions made above.

At this stage, I aim at producing an equation that will be able to represent a curve like the one I previously made using the actual values collected from the cooling of the cup of tea.

Newton's Law of Cooling states that: the rate of loss of heat from a body is directly proportional to the difference in the temperature of the body and its surroundings.

It can be expressed as: $\boldsymbol{\frac{dT}{dt}} = -\boldsymbol{k}(\boldsymbol{T} - \boldsymbol{T_s})$

Where its reads that: The rate of change of temperature with respect to time should be proportional to the difference between the temperature of the object and the ambient temperature.

Here:
T is the temperature of the cup of tea.
T_s is the temperature of the surrounding.
-K is the negative proportionality constant.

Why do we have a negative proportionality constant?

If $T \geq T_s$, then the temperature should decrease to meet that of the surrounding as stated above in the assumptions, meaning that cooling occurs. This will imply a negative instantaneous change hence, a negative proportionality constant.

7.1 Algebraically manipulate equation: $\frac{dT}{dt} = -k(T - T_s)$, to make all T and dT on one side (the subject terms).

Newton's equation of cooling, $\frac{dT}{dt} = -k(T - T_s)$, is my starting point from which I will derive my own equation for the cooling of my cup of tea.

Steps

- Multiply both sides by dt

$$\frac{dT}{dt} \times dt = -k(T - T_S) \times dt$$

- Divide both sides by T - T_s

$$\frac{dT}{dt} \times \frac{dt}{T - T_s} = \frac{-k(T - T_s)}{T - T_s} \times dt$$

This gives us: $dT \times \frac{1}{T - T_s} = -kdt$

- Integrate both sides.

$$\int dT \times \frac{1}{T - T_s} = -k \int dt$$

This gives us: $\ln|T - T_s| = -kt + C$

- Raise e (Euler's number) on both sides.

$$e^{\ln|T - T_S|} = e^{-kt+c}$$

This gives us: $|T - T_s| = e^{-kt+c}$

But e^{-kt+c} is the same as $e^{-kt} \times e^c$
Hence, we have: $|T - T_s| = e^{-kt}e^c$

Also, we know that T is hotter than T_s ($T \geq T_s$), this means that the net value is going to be positive. Therefore, as we want to look for the temperature of the cup of tea, we will make T the subject of the equation by adding T_s on both sides

This gives us: $\boldsymbol{T_{(t)} = e^{-kt}e^c + T_s}$

Let's summarize the equation and give meanings to each of the terms.

My final equation is: $\boldsymbol{T_{(t)} = Ae^{-kt} + T_s}$

Where:

$T_{(t)}$ is temperature of the cup of tea at time t
$A = e^c$ = the translation constant. This is related to the initial temperature rise of the cup of tea over the surrounding temperature.
-K is the negative proportionality constant.
t is time.
T_s is the temperature of the surrounding.

7.2 Solving for the value of A and k in the equation.

After giving a meaning to each of my terms, I will use simple algebra and simultaneous equation to find the value of A and k in the equation.

Step 1.
Re-arrange the equation using natural log to solve it more easily.

Given: $T_{(t)} = Ae^{-kt} + T_s$

- $T_{(t)} - T_s = Ae^{-kt}$
- $\ln\left(T_{(t)} - T_s\right) = (\ln A) + (-kt) \times \ln(e)$; where $\ln(e) = 1$
- $\ln\left(T_{(t)} - T_s\right) = \ln A - kt$

Step 2.
Insert the known values of the terms.

For: $\ln\left(T_{(t)} - T_s\right) = \ln A - kt$
At time (*t*) 0mins, tea temperature ($T_{(t)}$) is 68.0℃.
That is: *t = 0*, $T_{(0)} = 68.0℃$ *and* $T_s = 24.0℃$

Hence, we have:
$\ln(68.0 - 24.0) = \ln A - k(0)$
$\ln(44.0) = \ln A$
$e^{\ln(44.0)} = e^{\ln A}$
$A = 44.0$

Step 3.

Now that we have the value of A, let find the value of k.

When: $t = 5$, $T_{(5)} = 63.3℃$, $T_s = 24.0℃$ and $A = 44.0$

$$\ln(63.3 - 24.0) = \ln(44.0) - k(5)$$
$$\ln(39.3) = \ln(44.0) - 5k$$
$$k = \frac{\ln(39.3) - \ln(44.0)}{-5}$$
$$k = 0.0226$$

Producing the final Equation 1 by inserting the value of T_s, A and k in $T_{(t)} = Ae^{-kt} + T_s$ gives us:

$$\boldsymbol{T_{(t)} = 44.0e^{-0.0226t} + 24.0}$$

Note that all the calculated values are left in three significant figures because oversimplification can lead to a model that will not match the original curve.

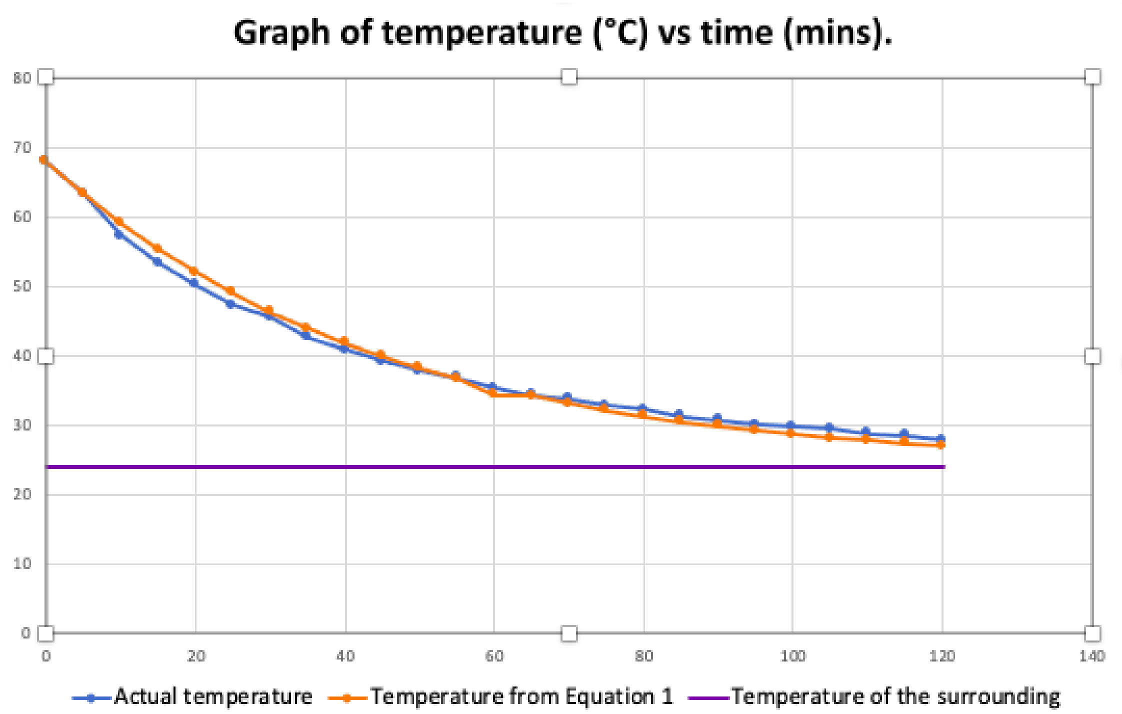
8. Collecting data for Equation 1.

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.												
Temp (℃)	68.0	63.3	59.1	55.3	51.9	49.0	46.3	43.9	41.8	39.9	38.2	36.7
Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55

Temperature Of Tea (℃) Over Two Hours In a Series Of Five Minutes Intervals.													
Temp (℃)	34.3	34.1	33.0	32.1	31.2	30.4	29.8	29.1	28.6	28.1	27.7	27.3	26.9
Time (mins)	60	65	70	75	80	85	90	95	100	105	110	115	120

Table 1.4: Tables of values for Equation 1.

9. Graph showing the relationship between time and actual temperature of the cup of tea, time and temperature gotten from Equation 1 and time and temperature of the surrounding (T_s)



Graph 1.1: Graph of the actual temperature, the temperature gotten from Equation 1 and the temperature of the surrounding vs time.

Comparing the quality of the actual values with the theorical prediction.

On the above graph, we can see that the curve of Equation one is almost identical to the initial curve of the cooling cup of tea. Both curves cross the y-axis at temperature 68℃ which appears to be the y-intercept and will never cross the horizontal asymptote of temperature 24℃ which is the room temperature.

But still, there are some slight difference between the curves that I will present on the table below.

Comparison of results			
Time(mins)	Actual values	Values of Equation 1	Differences between values
0	68.0	68.0	0.00
5	63.3	63.3	0.00
10	57.2	59.1	-1.90
15	53.3	55.3	-2.00
20	50.2	51.9	-1.70
25	47.2	49.0	-1.80
30	45.5	46.3	-0.80
35	42.7	43.9	-1.20
40	40.8	41.8	-1.00
45	39.2	39.9	-0.70
50	37.8	38.2	-0.40
55	36.8	36.7	0.10
60	35.3	34.3	1.00
65	34.3	34.1	0.20
70	33.7	33.0	0.70
75	32.8	32.1	0.70
80	32.2	31.2	1.00
85	31.3	30.4	0.90
90	30.7	29.8	0.90
95	30.0	29.1	0.90
100	29.7	28.6	1.10
105	29.5	28.1	1.40
110	28.8	27.7	1.10
115	28.5	27.3	1.20
120	27.8	26.9	0.90
Mean value of difference			0.024

Table 1.5: Comparison of the actual temperature and the values gotten from Equation 1.

10. Testing if the values gotten from Equation 1 are reliable using the standard deviation.

I will now test if the values gotten from Equation 1 are reliable by using the difference between the values (calculated above) and the mean value of difference to calculate the standard deviation.

x	$(x - \bar{x})$	$(x - \bar{x})^2$
68.0	-0.0240	5.76×10^{-4}
63.3	0.0240	5.76×10^{-4}
59.1	-1.924	3.701776
55.3	-2.024	4.096576
51.9	-1.724	2.972176
49.0	-1.824	3.326976
46.3	-0.824	0.678976
43.9	-1.224	1.498176
41.8	-1.024	1.048576
39.9	-0.724	0.524176
38.2	-0.424	0.179776
36.7	0.076	0.005776
34.3	0.976	0.952576
34.1	0.176	0.030976
33.0	0.676	0.456976
32.1	0.676	0.456976
31.2	0.976	0.952576
30.4	0.876	0.767376
29.8	0.876	0.767376
29.1	0.876	0.767376
28.6	1.076	1.157776
28.1	1.376	1.893376
27.7	1.076	1.157776
27.3	1.076	1.382976
26.9	0.876	0.767376

Calculating variance and standard deviation.

$$\sigma^2 = \frac{\sum(x - \bar{x})^2}{n} = \frac{28.748096}{25} = 1.14992384$$

$$\sqrt{\sigma^2} = \sqrt{1.14992384} = 1.072345019$$

Where:
$\bar{x}$ is the mean value of the difference between the actual values and the values of Equation 1.
x are the values gotten by subtracting Equation 1.
σ^2 is the variance.
$\sqrt{\sigma^2}$ is the standard deviation.

Standard deviation is a measure of how spread-out data values are around the mean. From the above value, we can see that the standard deviation is very low. This low standard deviation indicates that the data points tends to be very close to the mean. Hence, attesting the reliability of the values gotten from Equation 1 and its validity.

Note that I decided note to simplify the values obtained so that the value of the standard deviation can be exact and precise without being affected because of over simplification.

11. Limitations.

Despite all the attempts that I made to provide/bring out/come out with a realistic and reliable equation for the cooling of my cup of tea, there still are some sources of errors which were traits to the reliability of the equation and it usefulness to other.

- **Room temperature:** Firstly, the equation can only be used for a cup cooling in room whose temperature is 24°C. This is because the values of constant A and k in part 7.2 of the exploration where founded by substituting the room temperature and other data in the equation. Hence, making the equation to be applicable only in a room temperature of 24°C unless the value of A and k are changed according to the new room temperature.
- **The range of boiling water from 67 to 69 °C:** Also, to prepare the cup of tea for my exploration, I used a kettle to boil the water and the maximum temperature of boiling water from my kettle was in a range of 67°C to 69°C. This was a treat to my exploration because it does not reflect reality as normally, hot tea water boils over that range of temperature. Use a more effective kettle.
- **Reaction time:** Again, in the process of the exploration, I had to record the temperature of the cup of tea for every 5 minutes interval over two hours. The reactions time for every reading were different and the use of a thermometer made to work more complicated. Use a temperature sensor next time which will simply display the temperature of the cup of tea on the screen.
- **Insulation:** Moreover, the effect of insulation is another source of limitation to my equation as it does not take it into consideration meaning that, if another person uses my equation to calculate the time at which the tea will have a favorable temperature to be drink, my equation might not give accurate results because there exist mugs of different materials which has different insulation capacities.
- **Volume of water used:** Lastly, my equation does not take into account the volume of water used to make the cup of tea meanwhile volume and temperature are inversely proportional meaning that the larger the volume of water, the slower the rate at which it cools and the smaller the volume of water, the faster the rate at which it cools.

The above are possible source of errors to my exploration and limitations to my equation, nevertheless this exploration helped me to better understand and master the topic of: Calculus, Statistics, Algebra and Functions more than just being topics thoughts in a class room but also applicable in my daily life.

12. conclusion.

Now to answer my original question which is: after what amount of time my cup of tea will have reached the desired temperature for me to drink it, I will use Equation 1: $T_{(t)} = 44.0e^{-0.0226t} + 24.0$, to calculate that time assuming that 40.0℃ is my desired temperature to drink my tea.

This gives the following:

$$40.0 = 44.0e^{-0.0226t} + 24.0$$

$$44.0e^{-0.0226t} = 40.0 - 24.0$$

$$e^{-0.0226t} = \frac{40.0 - 24.0}{44.0}$$

$$\ln e^{-0.0226t} = \ln\left(\frac{40.0 - 24.0}{44.0}\right)$$

$$-0.0226t = \ln\left(\frac{40.0 - 24.0}{44.0}\right)$$

$$t = \ln\left(\frac{40.0 - 24.0}{44.0}\right) \Big/ -0.0226$$

$$t = 44.8\, minutes$$
$$t \cong 45.0\, minutes$$

This means that when I do my cup of tea, after 44.8 minutes my cup of tea will have a temperature of 40℃ and I will be able to drink it.

Overall, the process of measuring the temperature of my cooling cup of tea over two hours in a series of five minutes intervals was the one from which many systematic errors occurs reason why for the modelling of the graph, I decided to have three trials from which I used the mean value to model the equation. The process of making graph 1.0 and 1.1 help me to improve my graph making skills using Microsoft Excel tools. In part 7.1 and 7.2, I was able to learn how integration and differentiation are applicable in life and I also learnt how to work with natural logarithm. Part 10 was an important part of my exploration as I need to verify through the use of standard deviation if my equation was valid and the small value of standard deviation attested the validity of my equation. Nevertheless, if my equation was said to be not valid, it would have required me to look for another way to model my equation. Finally, despite the difficulties and limitations, my equation succeeded in producing an accurate model of my cooling cup of tea which is real and relevant in my life and can also be used by my family and friends to help them know when they can drink their cup of tea before it gets completely cold.

13. Citations.

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